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Note

Irreducible Groups Generated by Transvections over Finite Fields of Characteristic Two

Irreducible groups of linear transformations generated by transvections over a field of q elements have been completely classified for $q = 2$ by McLaughlin [9]. For q even, $q > 2$, and with additional hypotheses on the groups, classification theorems have been proved by Piper [10] and Wagner [11]. In this article, we show that if q is even, $q > 2$, and if the group in question is not the special linear group, then, with one exception, the transvections of the group are a conjugacy class of odd transpositions. Thus, we may use Aschbacher's classification of groups generated by odd transpositions [2]. Of course, a group generated by odd transpositions need not have an irreducible linear representation in which the odd transpositions are transvections. We show that, indeed, the Suzuki group $Sz(q)$ has no such representation over a finite field of characteristic two.

We assume throughout that V is a finite-dimensional vector space of dimension $n \geqslant 2$ over a field F of q elements, q even, $q > 2$. A linear transformation $T \neq 1$ of V is a *transvection* if $\mathrm{Ker}(T - 1)$ is a hyperplane and if $\mathrm{Im}(T - 1) \leqslant \mathrm{Ker}(T - 1)$. The hyperplane $H = \mathrm{Ker}(T - 1)$ is called the *axis* of T, and $P = \mathrm{Im}(T - 1)$ is a one-dimensional subspace called the *center* of T. If a linear group G contains a transvection with center P and axis H, then we say P is a *center for* G, H is an *axis for* G, and P is a *center for* H, H is an *axis for* P. The following useful facts are well known and/or easy to verify:

LEMMA 1. (i) *Let T be a linear transformation on V. T is a transvection with center P and axis H if and only if T has the form $T(v) = v + \lambda\varphi(v)a$ for all $v \in V$, where $\lambda \in F - \{0\}$, φ is a linear functional on V with kernel H and $\langle a\rangle = P \leqslant H$.* [1]

(ii) *If T is a transvection on V with center P and axis H, and if S is any nonsingular linear transformation on V, then STS^{-1} is a transvection with center $S(P)$ and axis $S(H)$.*

(iii) *The set of all transvections on V with a given center and axis, together with the identity, is a group isomorphic to the additive group of F.*

(iv) *If T, S are transvections on V with centers P, Q and axes H, K, respectively, then $TS = ST$ if and only if $P \leqslant K$ and $Q \leqslant H$.*

(v) *If T, S are distinct transvections on V, then TS is a transvection if and only if T and S have the same center or the same axis.*

(vi) *If the subspace $U \leqslant V$ is stable for a transvection with center P and axis H, then either $P \leqslant U$ or $U \leqslant H$.*

DEFINITION. If G is a finite group and D is a set of involutions in G such that $T, S \in D$ and $TS \neq ST$ imply that the order $|TS|$ of TS is odd, then D is a set of *odd transpositions* for G.

THEOREM 1. *Let V be a vector space of dimension $n \geqslant 2$ over a field F of q elements, q even, $q \geqslant 2$. Let G be a proper subgroup of $SL(V) = SL_n(q)$ that is irreducible and generated by the set D of transvections in G. Then, D is a conjugacy class of odd transpositions for G, unless $q = 4$, $\dim V = 3$ and $G/Z(G) \simeq A_6$.*

The proof of Theorem 1 depends on the following result of Piper:

THEOREM A [10, Main Theorem]. *If G is an irreducible proper subgroup of $SL(V)$, where V is a finite-dimensional vector space of dimension at least 2 over a field of q elements, q even, then every center P of G has a* unique *axis $P^{\perp}$, and every axis H of G has a* unique *center $H^{\perp}$, unless $\dim V = 3$, $q = 4$, and $G/Z(G) \simeq A_6$ or S_6.*

Only in the exceptional case $G/Z(G) \simeq A_6$ is G generated by transvections, and in this case, it is easy to check that the involutions in G are not odd transpositions.

A consequence of Piper's theorem is

LEMMA 2. *If G satisfies the hypotheses of Theorem 1, excluding the exceptional case $G/Z(G) \simeq A_6$, then*

(i) *$P \leqslant Q^{\perp}$ if and only if $Q \leqslant P^{\perp}$ for all centers P, Q for G.*

(ii) *If $T \in D$ and T has center P, then $C_D(T)$, the centralizer in D of T, is the set of all transvections in G whose axes contain P (or, dually, the set of all transvections in G whose centers lie on $P^{\perp}$).*

Proof. (i) Assume P and Q are distinct centers for G, and suppose $P \leqslant Q^{\perp}$ but $Q \nleqslant P^{\perp}$. Let $T, S \in D$ with centers P, Q, respectively. Then, a direct computation using Lemma 1(i) shows that $(TS)^2$ is a transvection with center P and axis $Q^{\perp}$, contradicting Theorem A. Now, (ii) is immediate from (i) and Lemma 1(iv).

Now we can prove Theorem 1. We assume that G is an irreducible proper subgroup of $SL(V)$ generated by transvections, and if $\dim V = 3$ and $q = 4$, we assume $G/Z(G) \not\simeq A_6$. Let T and S be noncommuting transvections in G

with centers P and Q, respectively. By Lemma 2, $P \nleqslant Q^{\perp}$ and $Q \nleqslant P^{\perp}$. Let $U = P^{\perp} \cap Q^{\perp}$, so $V = U \oplus P \oplus Q$. Suppose $| ST | = 2m$ for some m. Then, $(ST)^m$ is an involution, so $\mathrm{Im}[(ST)^m - 1] \leqslant \mathrm{Ker}[(ST)^m - 1]$ [5, I.3]. U is fixed pointwise by $(ST)^m$, so $U \leqslant \mathrm{Ker}[(ST)^m - 1]$. Suppose $U = \mathrm{Ker}[(ST)^m - 1]$. Let $x \in P \oplus Q$, $x \notin U$. Then, $(ST)^m x - x \in P \oplus Q$ since $P \oplus Q$ is stable for $(ST)^m$. But $(ST)^m x - x \in U$ also, so $(ST)^m x - x = 0$ and $x \in U$, which is a contradiction. Therefore, $\mathrm{Ker}[(ST)^m - 1]$ must be a hyperplane and $(ST)^m$ is a transvection. Let R be the center of $(ST)^m$. Since $S(ST)^m S = (TS)^m = (ST)^m = T(ST)^m T$, $(ST)^m$ commutes with S and with T. By Lemma 2, $R \leqslant P^{\perp} \cap Q^{\perp} = U$, so $R \nleqslant P \oplus Q$. But $P \oplus Q$ is stable for $(ST)^m$, so by Lemma 1(vi), $P \oplus Q \leqslant R^{\perp}$. Since $U \leqslant R^{\perp}$ also, we must have $(ST)^m = 1$, a contradiction. Therefore, $| ST |$ must be odd, and we have shown that transvections are odd transpositions.

Now we need to show that D is a conjugacy class of G. We already know that $D = D^G$, so suppose $D = D_1 \cup \cdots \cup D_k$, $k > 1$, a union of G-conjugacy classes. By another result of Piper [10, Lemma 3], since G is irreducible, G is transitive on its centers. Let P be any center of G, and let $S \in D_i$ have center Q. We know that $P = U(Q)$ for some $U \in G$, so by Lemma 1(ii), USU^{-1} is a transvection with center P and lies in $D_i^U = D_i$. Therefore, for any center P and for any i, $1 \leqslant i \leqslant k$, there is a transvection in D_i with center P. If G has only one center P, then P is stable for D and so for G. But since G is irreducible, $P = V$, which contradicts $\dim V \geqslant 2$. If $P \leqslant Q^{\perp}$ for every pair of centers P and Q for G, then again, P is stable for D, which is impossible. Hence, there exist centers P, Q for G with $P \nleqslant Q^{\perp}$. Choose $T \in D_i$ with center P and $S \in D_j$ with center Q, $i \neq j$. By Lemma 2, T and S do not commute, so TS has odd order, say $| TS | = 2m + 1$ for some m. But then, $S = (TS)^m T (ST)^m$ and T and S are conjugate, which contradicts $i \neq j$. Therefore, $k = 1$ and D is a conjugacy class. This completes the proof of Theorem 1.

Now we consider the question of which groups generated by a set D of odd transpositions actually have irreducible representations over a field of q elements with q even, $q > 2$, and where the elements of D are transvections. How does the abstract property of being generated by odd transpositions constrain the geometry of possible linear representations? In [2], Aschbacher associates with a group G generated by a set D of odd transpositions, a design $\mathscr{B}(D)$ with "points" the elements of D, with "blocks" the centralizers in D of elements of D, and with incidence being set-theoretic inclusion. The geometry of this design is related to the geometry of centers and axes for a representation of G as a group generated by transvections.

We will compute some of the parameters of $\mathscr{B}(D)$ when G satisfies the hypotheses of Theorem 1, excluding the exceptional case where $G/Z(G) \simeq A_6$. Let c be the number of centers for G, so c is also the number of axes for G.

Let d be the number of centers on an axis; since G is transitive on axes, this number is the same for every axis. By Lemma 2, d is also the number of axes on a center. For a center P of G, let t be the number of transvections with center P and axis $P^{\perp}$; by Lemma 1(ii) and the transitivity of G on centers, this number is the same for every P, $P^{\perp}$. Finally, for a center P, let l be the number of axes of G that contain exactly the centers on $P^{\perp}$. The main parameters of the design $\mathscr{B}(D)$ are the number v of points, the number b of blocks, the number k of points on a particular block, and the number r of blocks on a particular point. Clearly, for our group G, $v = tc$. For $T \in D$ with center P, the block $C_D(T)$ is the set of all transvections in G centered in $P^{\perp}$. If S is a transvection with center Q, then $C_D(T) = C_D(S)$ if and only if $P^{\perp}$ and $Q^{\perp}$ contain exactly the same centers, so the axes of G are in l-to-one correspondence with the blocks of $\mathscr{B}(D)$, and $lb = c$. The number of points on a block $C_D(T)$ is the number of transvections centered in the axis $P^{\perp}$ of T, so $k = td$. Given a point S with center Q, a block $C_D(T)$ lies on S if and only if Q is on the axis $P^{\perp}$ of T. The axes on Q are in l-to-one correspondence with the blocks on S, so $lr = d$. It follows that

$$v = tlb \qquad \text{and} \qquad k = tlr. \tag{1}$$

Choose an axis $P^{\perp}$ for G and let $P = P_1, \dots, P_d$ be the centers on $P^{\perp}$. Suppose the axes $P^{\perp} = Q_1^{\perp}, \dots, Q_l^{\perp}$ contain exactly the centers $P_1, \dots, P_d$. Then, for $i = 1, \dots, l$ and $j = 1, \dots, d$, $P_j \leqslant Q_i^{\perp}$ and $Q_i \leqslant P_j^{\perp}$. In particular, the Q_i are on $P^{\perp}$ and $l \leqslant d$ (as also follows from $lr = d$ above). Since the Q_i are among the P_j, we have $Q_i \leqslant Q_j^{\perp}$ for $i, j = 1, \dots, l$. For a center Q for G, let $D(Q)$ be the set of transvections in G with center Q. Then, the union of the $D(Q_i)$, $i = 1, \dots, l$, is a commuting set of lt transvections that generate an elementary abelian 2-subgroup E. Notice that if $Q_1, \dots, Q_u$ are independent and if $T_i \in D(Q_i)$, then $T_1, \dots, T_u = 1$ would imply a dependence relation among $Q_1, \dots, Q_u$, which is impossible. Therefore, E contains

$$\langle D(Q_1)\rangle \times \cdots \times \langle D(Q_u)\rangle.$$

By Lemma 1(iii), $\langle D(Q_i)\rangle$ is a group of order $t + 1$, so $(t + 1)^u \mid |E|$.

If a group generated by a set D of odd transpositions has a linear representation of this kind, then the parameters b, v, r, k for its design $\mathscr{B}(D)$ satisfy the relations (1), and the group contains an elementary abelian 2-subgroup E generated by tl commuting transvections.

Groups generated by odd transpositions for which the possibility of an irreducible representation generated by transvections would be of special interest are the Suzuki groups $Sz(q)$, q an odd power of 2, and the Fischer groups F_{22}, F_{23}, F_{24}. If $\dim V = 2$, the only groups generated by transvections on V are $SL(2, s)$, $s \mid |F|$, a group isomorphic to A_5 and a dihedral group [4, Chap. XII]. Wagner [11, Theorem 1.1] has proved that if dim $V = n \geqslant 3$, and if G is an irreducible group generated by transvections on V containing $t \geqslant 2$ transvections with center P and axis $P^{\perp}$, then G is isomorphic to one of $SL(n, t+1)$, $\mathrm{Sp}(n, t+1)$ or $U^{+}(n, (t+1)^2)$. Since $Sz(q)$, F_{22}, F_{23}, and F_{24} are isomorphic to none of these, we may assume $t = 1$ in any representation of the desired kind of any of these groups. This fact, together with only a portion of the information about the design $\mathscr{B}(D)$ for $Sz(q)$, yields

THEOREM 2. *Let $G = Sz(q)$, q an odd power of 2, $q > 2$. Let D be the conjugacy class of odd transpositions generating G. G has no linear representation as an irreducible group generated by the set D of transvections on a vector space V over a finite field of characteristic two.*

Proof. Since $Sz(q)$ is isomorphic neither to $SL(V)$ nor to A_6, any irreducible representation of $Sz(q)$ satisfying the hypotheses of Theorem 2 would also satisfy the hypotheses of Theorem 1. Other facts we need about $G = Sz(q)$ may be found in [7]. First, G has a single conjugacy class D of involutions [7, 2.8]. Further, for $u \in D$, $C_G(u)$ is the unique 2-Sylow subgroup of G containing u, and $C_D(u)$ is the set of $q - 1$ nontrivial elements in the center of $C_G(u)$ [7, 4.1, 4.2, 4.9]. Thus, each block of $\mathscr{B}(D)$ contains $k = q - 1$ points, and each point of $B(D)$ lies in a unique block, so $r = 1$. Now, relations (1) imply $tl = l = q - 1$, and G contains at least $q - 1$ commuting transvections, with distinct transvections having distinct centers. If T and S are two such commuting transvections, say with centers P and Q, respectively, then, either $T = S$ or TS is an involution and hence, also a transvection. By Lemma 1(v), if TS is a transvection, we must have $P = Q$ (and $P^{\perp} = Q^{\perp}$), which is a contradiction. Therefore, $T = S$ and $q - 1 = 1$, again a contradiction. This completes the proof of Theorem 2. (For other results on linear representations of $Sz(q)$, see [7, Sect. 10; 8].)

A detailed description of the design $\mathscr{B}(D)$ for D a class of 3-transpositions generating one of the Fischer groups F_{2j}, $j = 2, 3$, or 4, may yield information about possible linear representations of these groups as irreducible groups generated by the set D of transvections over finite fields of characteristic two. Arguing as for $G = Sz(q)$, we may assume that such a linear representation satisfies the hypotheses of Theorem 1. The number of transvections with a given center is $t = 1$, so the points of $\mathscr{B}(D)$ are in one-to-one correspondence with the centers for G. From [6, Theorem 2.1.3] and from the simplicity of the F_{2j}, we can conclude that for $T, S \in D$, $C_D(S) = C_D(T)$ if and only if $T = S$; so $l = 1$, and the blocks of $\mathscr{B}(D)$ are in one-to-one correspondence with the axes for G. Thus, the relations (1) become $v = b$ and $k = r$. Also, G contains a maximal commuting set L of $2j$ transvections, and $N = N_G(\langle L \rangle)$

acts on L like the Mathieu group M_{2j} [6]. Hence, N acts like M_{2j} on the $2j$ centers for L. Products of elements of L are trivial when the elements form a block in the Steiner system associated with the action of M_{2j} on L, and this implies dependence relations among the corresponding centers. We give an example of the kind of inference one can draw from this information. If $G = F_{24}$, then for distinct commuting transvections S and T with centers P and Q, respectively, there are at least 2^{10} transvections other than S and T commuting with S and T [3]. This implies that there are at least 2^{10} centers on $P^{\perp} \cap Q^{\perp}$, so $2^{10} \leqslant (q^{n-2} - 1)/(q - 1)$. Thus, $n \geqslant 4$, and if $n = 4$, $q \geqslant 2^{10}$.

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References

1. E. Artin, "Geometric Algebra," Interscience, New York, 1964.
2. M. Aschbacher, On finite groups generated by odd transpositions. I, *Math. Z.* **127** (1972), 45–56; Parts II–IV, *J. Algebra* **26** (1973), 451–491.
3. J. H. Conway, A construction for the smallest Fischer group F_{22}, *in* "Finite Groups '72," Proc. Gainsville Conference, North-Holland/American Elsevier, New York, 1973.
4. L. E. Dickson, "Linear Groups," Dover, New York, 1958.
5. J. Dieudonné, "La Géométrie des Groupes Classiques," 2nd ed. Springer–Verlag, Berlin, 1963.
6. B. Fischer, Finite groups generated by 3-transpositions I, *Invent. Math.* **13** (1971), 232–246.
7. H. Lüneberg, Die Suzukigruppen und ihre Geometrien, *in* "Lecture Notes in Mathematics **10**," Springer–Verlag, Berlin, 1965.
8. R. P. Martineau, On 2-modular representations of the Suzuki groups, *Amer. J. Math.* **94** (1962), 105–114.
9. J. E. McLaughlin, Some subgroups of $SL_n(\mathbf{F}_2)$, *Ill. J. Math.* **13** (1969), 108–115.
10. F. C. Piper, On elations of finite projective spaces of even order, *J. Lond. Math. Soc.* **43** (1968), 459–464.
11. A. Wagner, Groups generated by elations, *Abh. Math. Sem. Univ. Hamburg* **41** (1974), 190–205.

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